

PROBABILISTIC METHODS ON ERDOS PROBLEMS

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ABSTRACT. The following paper has been rewritten from [1] to make the ideas in graph theory more accessible.

The automorphism group of a graph is the group of isomorphisms that fix the adjacency matrix of the graph. We consider a parameter called average automorphism, the average of an automorphism group over a class of graphs, which can be used as a variable in a variety of contexts. For the set $\{G\}$ of (n, m) -graphs,

$$E(X|X < m) = m - \frac{[m-1][\binom{n}{2} - m + 1]}{[\binom{n}{2}] - 1}.$$

Let $|I|$ be the cardinality of a complete set of (n, m) -graphs up to isomorphism. If we are given $E(X|X < m)$, over a set of graphs $\{G\}$, we can extract a value $Aut_{av}(G)$:

$$E(X)n! = E(X|X < m)[n! - |Aut_{av}(G)|] + m|Aut_{av}(G)|,$$

and,

$$\begin{aligned} E(X)n!|\{L(I)\}| - m|Aut_{av}(G)||I| \\ = [E(X|X < m)][n!|I| - |Aut_{av}(G)|] \end{aligned}$$

allows us to find $|I|$. Notice the variable

$$Aut_{av}(G) = n!|I|/|\{L(I)\}|$$

for all isomorphism classes of graphs I .

1. COMPUTABLE TIME

The isomorphism problem is one of the 21 original *NP*-complete problems formulated by Karp. See [1]. When $m(G \cap H) = m(G)$, the two graphs are said to be isomorphic. There is an alternative, perhaps superior definition involving a map $f : V(G) \rightarrow V(H)$ such that $g_f(e = uv) = (f(u)f(v))$ has g_f a bijection from $g_f : E(G) \rightarrow E(H)$. The largest-size mutual placement of two graphs is said to be the intersection of the two graphs: $G \cap H$. The complement of G in the graph K_n is written $K_n - G$ is the graph induced by subtracting the entries of $A(G)$ from the entries of $A(K_n)$.

Some scheduling problems can be phrased on multigraphs. This often justifies the study of graph theory, but there are many kinds of scheduling problems including problems which can be solved by integer and linear programming. Cayley graphs are graph-representations of groups where the elements of the group form the vertex set, and then the edge set is determined by the action of the group presentation

on the vertex set. When an element in the generator of the presentation has order two, the 2-loop is generally considered a special undirected edge. The rest of the group elements form darts or directed arcs from v to $\alpha \circ v$.

It is clear that most problems in graph theory require more than linear time. That is, given a question and the order of the graph, it takes more than $n(G)$ steps in a computational algorithm to solve the question for an arbitrary graph. The k -SAT problem is an instance of 3-SAT and 3-SAT is NP-complete.

The validity of a truth assignment to a k -satisfiable statement can be checked in $k \times c$ where c is the number of literals. To see k -SAT is an instance of 3-SAT, take the logical negation of the statement $S_1 \subset k$ -SAT and expand each literal so that all of the clauses from each literal are taken together in some 3-tuple. The following ideas come from [3], but some of the treatment and proof is our own. Then take the logical negation of this expanded statement to get $S_2 \subset 3$ -SAT where S_1 and S_2 are logically equivalent. 3-SAT is an instance of 3-COL and 3-COL is NP-complete. Checking the validity of a 3-coloring takes at most n^2 steps algorithmically. To see 3-SAT is an instance of 3-COL, take a triangle colored in 3 colors and then two sets, which represent the separate clauses and their respective negations, and adjoin these partite sets to two of the colors in the triangle, one of which is held in common. We designate the color-pairs (1,2) and (0,2), for (true,2) and (false,2), according to which of the true or false assignments we want for the clauses. Iteratively attach the vertices of partite 3-tuples, (the 3-tuples represent the 3-clause literals), to the clauses they represent, vertex by vertex, as well as to the remaining vertex of the original triangle, colored color 2. Then form a cone over each of the 3-tuples and the 2-colored vertex from the original triangle; if the 3-tuple has a logical value 1, then the vertex cannot be colored a color i , $1 \leq i \leq 3$. That is, 3-SAT reduces to an instance of 3-COL where a negative answer for 3-satisfiability yields a 3-colorable graph and a positive answer for 3-satisfiability yields a k -colorable graph for $k > 3$. See [3]. The problem 3-COL is an instance of k -COL and k -COL is NP-complete. This is clear because given a valid k -coloring of a graph, the algorithm that checks the coloring is on the order of n^2 . To see k -COL is in P , observe that contracting the edge $G := G^e$ where e is the edge with $\min\{d(e) - |N(e)|\}$ always generates a largest K_t -minor of G . The algorithm that decides the edge of minimal $d(e) - |N(e)|$ takes $O(n^5)$ steps. There are at most $O(n^2)$ iterations of this algorithm. The algorithm for writing $A(G^e)$ from $A(G)$ takes $O(n^4)$ steps. It may become necessary to, without loss of generality, pick an edge incident a vertex of degree $\delta(G)$ if the contraction process reaches a tie.

There is a list of 21 problems that would establish the truth or falsity of $P = NP$. See [1]. Among these problems are the isomorphism problem, the knapsack problem, the Hamiltonian problem, the k -coloring problem and the k -SAT problem. We believe to have solved the first, third, and fourth problem with algorithms that run in polynomial time. There are several well-known solutions to the Hamiltonian problem that run in polynomial time, or, rather, poly-time, P . The isomorphism question can be solved by taking breadth-first searches of the two graphs at each vertex of the respective vertex sets and establishing criteria for which a 1-1 matching between the breadth-first permutations demonstrates isomorphism. Suppose there is a 1-1 matching that does not establish isomorphism and perform induction to obtain a minimal counterexample to the contrary claim. Delete a vertex of minimal

degree and demonstrate the isomorphism on the graphs. (The sets of vertex-deleted trees are identical and there is a base case.) Now there are two possibilities: (1) Either the deleted vertex could map to another identical vertex under breadth-first search, or (2) The deleted vertex does not establish a new isomorphism and still preserves the original breadth-first search trees. (1a): Suppose the image of the deleted vertex v , call it vertex w , has $v \sim w$. Then delete v and w and notice that we did not have a minimal counterexample. (1b): By the reasoning in (1a), $v \not\sim w$. So if we eliminate $\{v, w\}$ from $V(G)$ the two graphs G and H are isomorphic. When we re-assemble the two graphs, it becomes clear upon induction that the non-cliqued vertices from $N(v) \cup N(w)$ are distinct and unique up to choice of one vertex from outside the triangular difference of the two neighborhoods. (2): The vertex-deleted trees must all fall into an orbit. The tree interchanged with the vertex-deleted trees falls into the same orbit as well, under the correct inductive assumption. From the proof, above, it holds that the Reconstruction conjecture holds for trees and so it holds for all graphs. We could send permutations for each vertex and then reassemble graphs using a catalog. We believe that the satisfiability problem is equivalent to a special version of an edge-labeling conjecture. Let G have $E(G) = \{uv : u \wedge v \in \Gamma_i\}$ where Γ_i is a literal and $i \in [k]$. The graph G represents a satisfiable statement precisely when the parity of the sum of the appearances of the clauses assigned truth value 1 is odd.

If we delete the vertices that are assigned truth values 0: $\{F_i\}$, to form G_{sat} , then the vertex sets (literals) $V(\Gamma_i - F_i) = V(T_i)$ for $i = t$ that have odd order induced by each such literal have even degree. That is, an odd number of the subgraphs induced by the $\{T_i\}$, $[V(T_i)]$ are not Eulerian when G_{sat} represents a satisfactory truth assignment. (If G represents a satisfiable statement, the number of literals with an odd number of clauses assigned truth value 1 is odd and is,

$$\sum_{\Gamma} T_i \bmod 2 = 1.)$$

This parity occurs when either, for each statement, $v \in V(G)$, *either* v or $\neg v$ (the negation of v) is included in an odd number of literals Γ . If and only if G has an Eulerian subgraph on some truth assignment of $(0, 1)$ to the clauses of $V(G)$ does some G_{sat} -subgraph of G represent a satisfactory truth assignment. Such an Eulerian subgraph exists if and only if the number of appearances of v and $\neg v$ is odd across the various literals. Form G^* by $H : V(G) \rightarrow V(G^*)$ where

$$H : \{v, \neg v\} \rightarrow v^*.$$

If and only if G has all vertices of odd degree is G a satisfiable statement.

2. ELLIPTIC EQUATIONS

Our main contention here in this work is that based on previous work we can say

$$Z = \frac{y^2 - d}{aZ + c}$$

and let $aZ^2 + cZ = y^2 - d$ to give the following quadratic equation listed below

$$x = \pm \sqrt{\frac{-c \pm \sqrt{c^2 + y^2 - 4d}}{2a}}.$$

Invoke the Banach Contraction Theorem: Then $X = f(X) = P(X)/Q(X)$ becomes

$$\frac{y^2 - d}{ax^2 + c} = \frac{y^2 - d}{a[\frac{y^2 - d}{ax^2 + c}]^2 + c}.$$

Then we can use the last expression to get a one-step continued fraction based on the convergence of $f(X)$ to X :

$$\frac{y^2 - d}{aZ + c} = \frac{y^2 - d}{a[\frac{y^2 - d}{aZ + c}] + c}.$$

Here we are discussing equations of the form

$$ax^3 + cx + d = y^2.$$

The equation is said to be elliptic and part of the theory of elliptic equations is to determine genus and sketch the curve, including rational points. Let the discriminant of the last quadratic equal $D_0 = \sqrt{c^2 + y^2 - 4d}$. Then $D_0^2 - y^2 = c^2 - 4d$. The expression gives $(D_0 - y)(D_0 + y) = c^2 - 4d = st$. Letting $D_0 - y = s$ and $D_0 + y = t$ gives $g(c^2 - 4d)$ solutions for D_0 where $g \geq 1$ with equality in the case that $c^2 - 4d$ has a prime numerator and denominator and $2g$ is defined to be the number of divisors of $c^2 - 4d$. However, it is still required that x is rational. So one of the solutions for D_0 must satisfy $x^2 = (-c \pm D_0)(2a)^{-1}$ in order for (x, y) to be a rational pair. However, $D_0 = \frac{s-t}{2}$.

First, let us consider the following quadratic based on an arbitrary factorization of the discriminant

$$x^2 = \frac{-2c \pm (s - t)}{4a}.$$

Then, without loss of generality, let us solve for the variable x by computing

$$x = \sqrt{\frac{4d + 2ct + t^2 - c^2}{4t}}.$$

So then, let $c^2 - 4d$ be arbitrary. Pick $c^2 - 4d + W^2 = t(t + 2c)$. Now, produce the value of the dummy variable W as the value $W = -(2d/c)$. Use the factorization technique again and let $t(t + 2c) = [(c^2 - 2d)/rc][r(c^2 + 2d)/c]$. Finally, let $[(r^2 - 1)2d] = [(r^2 - 2r - 1)c^2] \rightarrow 2d/c^2 = (r^2 - 1)/(r^2 - 2r - 1)$. Whenever r is rational, we have generated a rational (x, y) -pairs of multiplicity 4 and some (perhaps 1) of the solutions are extraneous. We have $t = f(r)$ and $(x, y) = g(f(r))$.

A gentler way to approach the problem is to use the Fundamental Theorem of Algebra and reason that

$$x = b, \frac{b \pm \sqrt{b^2 - 4c + 4y^2}}{2}$$

and that the solutions are all rational when $\sqrt{b^2 - 4c + 4y^2}$ is rational. We speculate that both methods generate the same answers. In all cases, the problem reduces to the number of rational identities of the form $(W - 2y)(W + 2y) = b^2 - 4c$.

REFERENCES

- [1] J. Gilbert. Probabilistic Methods on Erdos Problems. ArXiv, 1107.3279v3.